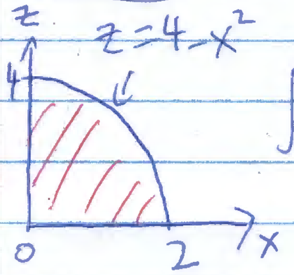
Solution for HW6

Ex 15.5: 44) $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx = \int_0^2\int_0^{4-x^2}\frac{\sin 2z}{4-z}\,x\,dz\,dx =$



$$\int_0^4\int_0^{\sqrt{4-z}} \frac{\sin 2z}{4-z}\,x\,dx\,dz = \frac12\int_0^4 \sin 2z\,dz = \frac{1-\cos 8}{4} \quad \left(\text{or } \frac{\sin^2 4}{2}\right)$$

Ex 15.6: 28)

$$I_L = \int_{-2}^2\int_{-2}^4\int_{-1}^{\frac{2-y}{2}} \left[(x-4)^2+y^2\right]dz\,dy\,dx$$

$$= \frac12\int_{-2}^2\int_{-2}^4 (x^2-8x+16-y^2)(4-y)\,dy\,dx$$

$$= \frac12\int_{-2}^2\int_{-2}^4 (4x^2-32x+64-4y^2-x^2y+8xy-16y-y^3)\,dy\,dx$$

$$= \int_{-2}^2 (9x^2-72x+162)\,dx$$

$$= 696$$

30) a)

$$M = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{x}} kxy(4-x^2)\,dy\,dx$$

$$= \frac12\int_0^2 kx^2(4-x^2)\,dx = \frac{32k}{15}$$

b)

$$M_{yz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kx^2y\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{x}} kx^2y(4-x^2)\,dy\,dx$$

$$= \frac12\int_0^2 kx^3(4-x^2)\,dx = \frac{8k}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac54$$

$$M_{xz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy^2\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{x}} kxy^2(4-x^2)\,dy\,dx$$

$$= \frac13\int_0^2 kx^{\frac52}(4-x^2)\,dx = \frac{256\sqrt2}{231}k \Rightarrow \bar{y} = \frac{M_{xz}}{M} = \frac{40\sqrt2}{77}$$

$$M_{xy} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxyz\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{x}} kxy\frac{(4-x^2)^2}{2}\,dy\,dx$$

$$= \frac14\int_0^2 kx^2(4-x^2)^2\,dx = \frac{256}{105}k \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac87$$

36) By Parallel Axis Theorem, $I_L = I_{c.m.} + mh^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$ //

Ex 15.7: 12) a) $\int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta$ //

b) $\int_0^{2\pi}\int_0^1\int_0^z r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\,dr\,dz\,d\theta$ //

c) $\int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\,d\theta\,dz\,dr$ //

14) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_0^{r\cos\theta} r^3\,dz\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\,dr\,d\theta$

$= \frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta = \frac{2}{5}$ //

38) a) $V = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi$

$= (2\pi)\left(\frac{8}{3}\right)\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin\phi\,d\phi = \frac{8\pi}{3}$ //

40) a) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{3}{\sqrt{2}}}\int_r^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$ //

b) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^3 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ //

c) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^3 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{\pi}{2}\int_0^{\frac{\pi}{4}}\int_0^3 \rho^2\sin\phi\,d\rho\,d\phi$

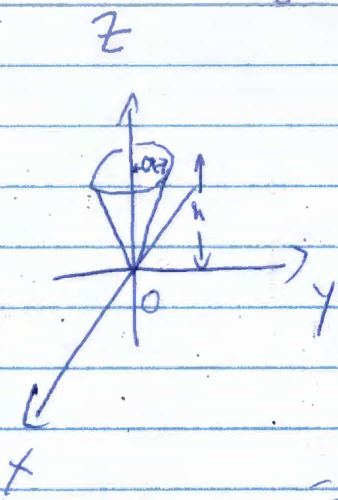
$= \left(\frac{\pi}{2}\right)(9)\int_0^{\frac{\pi}{4}}\sin\phi\,d\phi = \frac{9\pi(2-\sqrt{2})}{4}$ //

62) $\begin{cases} x^2+y^2+z^2 = 2 \\ z = x^2+y^2 \end{cases} \Rightarrow z^2+z-2=0 \Rightarrow z=1$ or -2 (rejected)

$$\therefore V = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = 2\pi\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr$$

$$= 2\pi\int_0^1 r\left(\sqrt{2-r^2}-r^2\right)dr = \frac{\pi(8\sqrt{2}-7)}{6}$$

80) Let a and h be the base radius and height of the cone respectively.



Put the cone in $\mathbb{R}^3$ s.t. the axis of symmetry is the z-axis and the vertex of the cone is $(0,0,0)$.

Then $M = \frac{\pi a^2 h}{3}$ and

$$M_{xy} = \int_0^{2\pi}\int_0^a\int_{(\frac{h}{a})r}^{h} zr\,dz\,dr\,d\theta = 2\pi\int_0^a\int_{(\frac{h}{a})r}^{h} zr\,dz\,dr$$

$$= (2\pi)\left(\frac{1}{2}\right)\int_0^a\left(h^2-\frac{h^2}{a^2}\right)r\,dr = \frac{h^2a^2\pi}{4}$$

$$\therefore \bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}h.$$

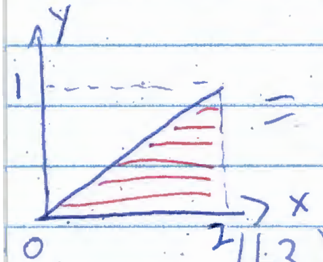
By symmetry, $\bar{x} = 0 = \bar{y}$.

$\Rightarrow$ The centroid is $\frac{1}{4}$ of the way from the base to the vertex.

Practice Problems:

Ex 15.5: 41)

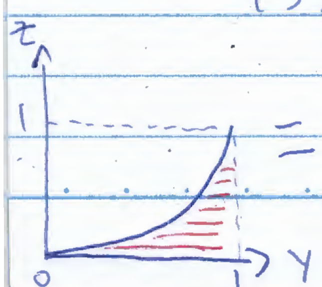
$$\int_0^4\int_0^1\int_{2y}^{2}\frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz = \int_0^1\int_0^2\int_0^{\frac{x}{2}}\frac{4\cos(x^2)}{2\sqrt{z}}\,dy\,dx\,dz$$



$$= \int_0^4\int_0^2\frac{\cos(x^2)}{z}\,x\,dx\,dz = \frac{\sin 4}{2}\int_0^4 z^{-\frac{1}{2}}\,dz = 2\sin 4$$

43)

$$\int_0^1\int_{z^{\frac{1}{3}}}^1\int_0^{\ln 3}\frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,dx\,dy\,dz = 4\int_0^1\int_{z^{\frac{1}{3}}}^1\frac{\pi\sin(\pi y^2)}{y^2}\,dy\,dz$$



$$= 4\int_0^1\int_0^{y^3}\frac{\pi\sin(\pi y^2)}{y^2}\,dz\,dy = 4\int_0^1\sin(\pi y^2)(\pi y)\,dy = 4$$

Ex 15.6: 35) a) WLOG, assume that the centre of mass is placed in yz-plane. Then $\bar{x}=0$ and $M_{yz}=\bar{x}M=0$.

b) $I_L=\iiint_D |v-h\hat{i}|\,dm=\iiint_D (x^2-2xh+h^2+y^2)\,dm$

$=\iiint_D (x^2+y^2)\,dm-2h\iiint_D x\,dm+h^2\iiint_D dm$

$=I_{c.m}-2hM_{yz}+h^2m$

$=I_{c.m.}+mh^2$ //

Ex 15.7: 35) $V=\int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=2\pi\int_0^{\pi}\int_0^{1+\cos\phi}\rho^2\sin\phi\,d\rho$

$=\frac{2\pi}{3}\int_0^{\pi}(1+\cos\phi)^3\sin\phi\,d\phi=\frac{-2\pi}{3}\int_0^{\pi}(1+\cos\phi)^3\,d(1+\cos\phi)$

$=-\frac{2\pi}{3}\left[\frac{1}{4}(1+\cos\phi)^4\right]_0^{\pi}=\frac{8\pi}{3}$ //

37) $V=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi$

$=2\pi\left(\frac{8}{3}\right)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^3\phi\sin\phi\,d\phi=\frac{-16\pi}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^3\phi\,d\cos\phi$

$=\frac{\pi}{3}$ //

81) Let $\delta(r)=kr+c$ for some $k\neq 0, c\in\mathbb{R}$.
As $\delta(R)=0$, we have $c=-kR$.
$\therefore\ \delta(r)=k(r-R)$.
Note that $M=\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho^2k(\rho-R)\sin\phi\,d\rho\,d\phi\,d\theta$

$=2\pi\int_0^{\pi}\int_0^{R}\rho^2k(\rho-R)\sin\phi\,d\rho\,d\phi=2\pi\int_0^{\pi}\left(\frac{kR^4}{4}-\frac{kR^4}{3}\right)\sin\phi\,d\phi$

$$= (2\pi)\left(\frac{-kR^4}{12}\right)(2) = \frac{-k\pi R^4}{3}$$

$$\therefore k = \frac{-3M}{\pi R^4} \quad \text{and} \quad \delta(b) = -kR = \frac{3M}{\pi R^3}$$

85) The equation $r=f(z)$ implies that $(f(z), z, \theta)$ lies on the surface $\forall\theta$. So the surface is symmetric along z-axis.

86) The equation $\rho = f(\phi)$ implies that $(f(\phi), \phi, \theta)$ lies on the surface $\forall\theta$. So the surface is symmetric along z-axis.